

# What is typical?

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## Abstract

Let  $\xi$  be a random measure on a locally compact second countable topological group and let  $X$  be a random element in a measurable space on which the group acts. In the compact case, we give a natural definition of the concept that the origin is a typical location for  $X$  in the mass of  $\xi$ , and prove that when this holds the same is true on sets placed uniformly at random around the origin. This new result motivates an extension of the concept of typicality to the locally compact case where it coincides with the concept of mass-stationarity. We describe recent developments in Palm theory where these ideas play a central role.

*Keywords:* random measure; typical location; Poisson process; point-stationarity; mass-stationarity; Palm measure; allocation; invariant transport.

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## 1 Introduction

The word ‘typical’ is sometimes used in probability contexts in an informal way. For instance, a typical element in a finite set, – or in a finite interval, – is usually interpreted as an element chosen according to the uniform distribution. Also, after adding a point at the origin to a stationary Poisson process, the new point is often referred to as a typical point of the process. Note that in both these examples the choice of an element (point) is far from being arbitrary; typical does not mean arbitrary. In this paper we attempt to make the term ‘typical’ precise.

We consider a random measure  $\xi$  on a locally compact second countable topological group and a random element  $X$  in a measurable space on which the group acts. In the compact case, we give a natural definition of the concept that the origin is a typical location for  $X$  in the mass of  $\xi$ , and prove that this property is equivalent to the more mysterious property that the same is true on sets placed uniformly at random around the origin. This new result motivates an extension of the concept of typicality to the locally compact case where it coincides with the concept of mass-stationarity which was introduced in [8]. We then outline recent developments in Palm theory of stationary random measures where these concepts play a central role.

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## 2 Preliminaries

Let  $G$  be a locally compact second countable topological group equipped with the Borel  $\sigma$ -algebra  $\mathcal{G}$ . Then the mapping from  $G \times G$  to  $G$  taking  $(s, t)$  to  $st$  and the mapping from  $G$  to  $G$  taking  $t$  to  $t^{-1}$  are measurable. We refer to the neutral element  $e$  of  $G$  as the *origin* and to the elements of  $G$  as *locations*.

For a measure  $\mu$  on  $(G, \mathcal{G})$  and a set  $C \in \mathcal{G}$  such that  $0 < \mu(C) < \infty$ , define the conditional probability measure  $\mu(\cdot | C)$  by

$$\mu(A | C) = \mu(A \cap C) / \mu(C), \quad A \in \mathcal{G}.$$

For convenience, we let  $\mu(\cdot | C)$  equal some fixed probability measure if  $\mu(C) = 0$ . For  $t \in G$ , let  $t\mu$  be the pushforward of  $\mu$  under the mapping  $s \mapsto ts$ , that is,

$$t\mu(A) := \mu(t^{-1}A), \quad A \in \mathcal{G}.$$

Let  $\lambda \neq 0$  be a left-invariant Haar measure, see e.g. Theorem 2.27 in [6]. An example is any countable group  $G$  with  $\lambda$  the counting measure. Another example is  $\mathbb{R}^d$  under addition with  $\lambda$  the Lebesgue measure.

Let  $\stackrel{D}{=}$  denote identity in distribution. Let  $\xi$  be a nontrivial random measure on  $(G, \mathcal{G})$ . Say that  $\xi$  is *stationary* if

$$t\xi \stackrel{D}{=} \xi, \quad t \in G.$$

Let  $G$  act on a measurable space  $(E, \mathcal{E})$  measurably, that is, such that the mapping from  $G \times E$  to  $E$  taking  $(t, x)$  to  $tx$  is measurable. Let  $X$  be a random element in  $(E, \mathcal{E})$ . For instance,  $X$  could be a random field  $X = (X_s)_{s \in G}$  and  $tX = (X_{t^{-1}s})_{s \in G}$  for  $t \in G$ . Say that  $X$  is *stationary* if

$$tX \stackrel{D}{=} X, \quad t \in G. \tag{2.1}$$

Put  $t(X, \xi) = (tX, t\xi)$ . Say that  $(X, \xi)$  is *stationary* if

$$t(X, \xi) \stackrel{D}{=} (X, \xi), \quad t \in G.$$

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be the probability space on which the random elements in this paper are defined. If  $S$  is a random element in  $(G, \mathcal{G})$ , let  $S^{-1}$  denote the group inverse of  $S$  (and not the inverse of  $S$  as a function defined on  $\Omega$ ).

## 3 Compact groups and typicality

In this section assume that  $G$  is compact. Then both  $\lambda$  and  $\xi$  are finite and  $\lambda$  is also right invariant (see e.g. Theorem 2.27 in [6]). An example is any finite group with  $\lambda$  the counting measure. Another example is the  $d$ -dimensional rotation group.

Let  $S$  be a random element in  $(G, \mathcal{G})$ . Say that  $S$  is uniformly distributed on  $C \in \mathcal{G}$  if  $S$  has the distribution  $\lambda(\cdot | C)$ . Note that  $\lambda(\cdot | G) = \lambda / \lambda(G)$ .

**Definition 3.1.** (a) If  $S$  is uniformly distributed on  $G$ , then  $S$  is a *typical* location in  $G$ .

(b) If  $S$  is a typical location in  $G$  and independent of  $X$ , then  $S$  is a typical location for  $X$ .

(c) If  $S$  is a typical location for  $X$  and  $S^{-1}X \stackrel{D}{=} X$ , then the *origin* is a typical location for  $X$ .

**Theorem 3.2.** Let  $G$  be compact.

(a) If  $S$  is a typical location for  $X$ , then  $S^{-1}X$  is stationary.

(b) The origin is a typical location for  $X$  if and only if  $X$  is stationary.

*Proof.* (a) If  $S$  is a typical location for  $X$  then so is  $St^{-1}$  for each  $t \in G$ . Thus  $(St^{-1})^{-1}X$  has the same distribution as  $S^{-1}X$ . But  $(St^{-1})^{-1}X = t(S^{-1}X)$ . Thus  $S^{-1}X$  is stationary.

(b) Let  $S$  be a typical location for  $X$ . If  $S^{-1}X \stackrel{D}{=} X$  then  $X$  is stationary since  $S^{-1}X$  is stationary. Conversely, if  $X$  is stationary then  $S^{-1}X \stackrel{D}{=} X$  follows from (2.1) and the independence of  $S$  and  $X$ .  $\square$

We shall now extend the above typicality concepts from the uniform distribution to random measures.

**Definition 3.3.** (a) If the conditional distribution of  $S$  given  $\xi$  is  $\xi(\cdot | G)$ , then  $S$  is a *typical* location in the *mass* of  $\xi$ .

(b) If  $S$  is a typical location in the mass of  $\xi$  and  $S^{-1}\xi \stackrel{D}{=} \xi$ , then the *origin* is a typical location in the mass of  $\xi$ .

(c) If  $S$  is a typical location in the mass of  $\xi$  and conditionally independent of  $X$  given  $\xi$ , then  $S$  is a typical location for  $X$  in the mass of  $\xi$ .

(d) If  $S$  is a typical location for  $X$  in the mass of  $\xi$  and  $S^{-1}(X, \xi) \stackrel{D}{=} (X, \xi)$ , then the *origin* is a typical location for  $X$  in the mass of  $\xi$ .

The following theorem says that the origin is a typical location for  $X$  in the mass of  $\xi$  if and only if it is a typical location for  $X$  in the mass of  $\xi$  on *sets placed uniformly at random around the origin*.

**Theorem 3.4.** Let  $G$  be compact. Then the origin is a typical location for  $X$  in the mass of  $\xi$  if and only if for all  $C \in \mathcal{G}$  such that  $\lambda(C) > 0$

$$(V_C^{-1}(X, \xi), U_C V_C) \stackrel{D}{=} ((X, \xi), U_C) \quad (3.1)$$

where

(i)  $U_C$  is uniformly distributed on  $C$  and independent of  $(X, \xi)$ , and

(ii)  $V_C$  has the conditional distribution  $\xi(\cdot | U_C^{-1}C)$  given  $(X, \xi, U_C)$ .

*Proof.* Suppose (3.1) holds for all  $C$ . Then in particular  $V_G^{-1}(X, \xi) \stackrel{D}{=} (X, \xi)$ . Moreover, since  $U_G^{-1}G = G$  we have from (ii) that  $V_G$  has the conditional distribution  $\xi(\cdot | G)$  given  $(X, \xi, U_G)$ . This implies that  $V_G$  is a typical location in the mass of  $\xi$  and also that  $V_G$  is conditionally independent of  $X$  given  $\xi$ . Thus the origin is a typical location for  $X$  in the mass of  $\xi$ .

Conversely, suppose the origin is a typical location for  $X$  in the mass of  $\xi$ . For nonnegative measurable  $f$  and with  $U_C$  and  $V_C$  as above we have

$$\mathbb{E}[f(V_C^{-1}(X, \xi), U_C V_C)] = \mathbb{E}\left[\iint 1_{\{u \in C\}} 1_{\{v \in u^{-1}C\}} f(v^{-1}(X, \xi), uv) \frac{\xi(dv)}{\xi(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)}\right].$$

Let  $S$  be a typical location for  $X$  in the mass of  $\xi$ . Then we obtain

$$\begin{aligned} & \mathbb{E}[f(V_C^{-1}(X, \xi), U_C V_C)] \\ &= \mathbb{E}\left[\iint 1_{\{u \in C\}} 1_{\{v \in u^{-1}C\}} f(v^{-1}S^{-1}(X, \xi), uv) \frac{(S^{-1}\xi)(dv)}{(S^{-1}\xi)(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)}\right] \\ &= \mathbb{E}\left[\iint 1_{\{u \in C\}} 1_{\{S^{-1}v \in u^{-1}C\}} f((S^{-1}v)^{-1}S^{-1}(X, \xi), uS^{-1}v) \frac{\xi(dv)}{(S^{-1}\xi)(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)}\right] \\ &= \mathbb{E}\left[\iiint 1_{\{u \in C\}} 1_{\{s^{-1}v \in u^{-1}C\}} f(v^{-1}(X, \xi), us^{-1}v) \frac{\xi(dv)}{(s^{-1}\xi)(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)} \frac{\xi(ds)}{\xi(G)}\right]. \end{aligned}$$

Make the variable substitution  $r = us^{-1}v$  (equivalently,  $u = rv^{-1}s$ ) and use right-invariance of  $\lambda$  to obtain

$$\begin{aligned} & \mathbb{E}[f(V_C^{-1}(X, \xi), U_C V_C)] \\ &= \mathbb{E}\left[\iiint 1_{\{v^{-1}s \in r^{-1}C\}} 1_{\{r \in C\}} f(v^{-1}(X, \xi), r) \frac{\xi(dv)}{(v^{-1}\xi)(r^{-1}C)} \frac{\lambda(dr)}{\lambda(C)} \frac{\xi(ds)}{\xi(G)}\right] \\ &= \mathbb{E}\left[\iiint 1_{\{s \in r^{-1}C\}} 1_{\{r \in C\}} f(v^{-1}(X, \xi), r) \frac{\xi(dv)}{v^{-1}\xi(r^{-1}C)} \frac{\lambda(dr)}{\lambda(C)} \frac{v^{-1}\xi(ds)}{\xi(G)}\right] \\ &= \mathbb{E}\left[\iint 1_{\{s \in r^{-1}C\}} 1_{\{r \in C\}} f(S^{-1}(X, \xi), r) \frac{(S^{-1}\xi)(ds)}{(S^{-1}\xi)(r^{-1}C)} \frac{\lambda(dr)}{\lambda(C)}\right]. \end{aligned}$$

Again, apply the fact that  $S$  is a typical location for  $X$  in the mass of  $\xi$  (and recall we are assuming that the origin is a typical location for  $X$  in the mass of  $\xi$ ) to obtain

$$\begin{aligned} \mathbb{E}[f(V_C^{-1}(X, \xi), U_C V_C)] &= \mathbb{E}\left[\iint 1_{\{s \in r^{-1}C\}} 1_{\{r \in C\}} f((X, \xi), r) \frac{\xi(ds)}{\xi(r^{-1}C)} \frac{\lambda(dr)}{\lambda(C)}\right] \\ &= \mathbb{E}\left[\int \left(\int 1_{\{s \in r^{-1}C\}} \frac{\xi(ds)}{\xi(r^{-1}C)}\right) 1_{\{r \in C\}} f((X, \xi), r) \frac{\lambda(dr)}{\lambda(C)}\right] \\ &= \mathbb{E}\left[\int 1_{\{r \in C\}} f((X, \xi), r) \frac{\lambda(dr)}{\lambda(C)}\right] \\ &= \mathbb{E}[f((X, \xi), U_C)] \end{aligned}$$

that is, (3.1) holds. In the above calculation expressions like  $((s^{-1}\xi)(u^{-1}C))^{-1}$  can be given some fixed (arbitrary) value if  $(s^{-1}\xi)(u^{-1}C) = 0$ . This requires some care but can be accomplished as in the first part of the proof of Theorem 6.3 in [8].  $\square$

## 4 Locally compact groups, typicality and mass-stationarity

We shall now drop the condition that  $G$  is compact. Then  $\lambda$  and  $\xi$  are only  $\sigma$ -finite so Definitions 3.1 and 3.3 do not work. However, Theorem 3.4 suggests a way to define

typicality of the origin in this case: demand that the origin is a typical location for  $X$  in the mass of  $\xi$  on sets placed uniformly at random around the origin.

**Definition 4.1.** (a) If (3.1) holds for all relatively compact  $\lambda$ -continuity sets  $C$  with  $\lambda(C) > 0$ , then the *origin* is a *typical* location for  $X$  in the *mass* of  $\xi$ .

(b) If (3.1) holds with  $X$  deleted, then we say that the origin is a typical location in the mass of  $\xi$ .

(c) If (a) is true with  $\xi = \lambda$ , then we say that the origin is a typical location for  $X$ .

The reason we choose here to restrict  $C$  to be a  $\lambda$ -continuity set (that is, a set with boundary having  $\lambda$ -measure zero) is that then the property in the definition is exactly the property used in [8] to define *mass-stationarity*:  $(X, \xi)$  is called mass-stationary if the origin is a typical location for  $X$  in the mass of  $\xi$  in the sense of Definition 4.1.

Now recall (see e.g. [6] for the case  $G = \mathbb{R}^d$  and [7] for the general case) that a pair  $(X, \xi)$  is called a *Palm version* of a stationary pair  $(Y, \eta)$  if for all nonnegative measurable functions  $f$  and all compact  $A \in \mathcal{G}$  with  $\lambda(A) > 0$ ,

$$\mathbb{E}[f(X, \xi)] = \mathbb{E}\left[\int_A f(t^{-1}(Y, \eta))\eta(dt)\right] / \lambda(A). \quad (4.1)$$

In this definition  $(X, \xi)$  and  $(Y, \eta)$  are allowed to have distributions that are only  $\sigma$ -finite and not necessarily probability measures. The distribution of  $(X, \xi)$  is finite if and only if  $\eta$  has finite intensity, that is, if and only if  $\mathbb{E}[\eta(A)] < \infty$  for compact  $A$ . In this case the distribution of  $(X, \xi)$  can be normalized to a probability measure.

The following equivalence of mass-stationarity and Palm versions was established in [8] in the Abelian case and extended to the non-Abelian case in [7].

**Theorem 4.2.** *Let  $G$  be locally compact and allow the distributions of  $(X, \xi)$  and  $(Y, \eta)$  to be only  $\sigma$ -finite. Then  $(X, \xi)$  is mass-stationary (that is, the origin is a typical location for  $X$  in the mass of  $\xi$ ) if and only if  $(X, \xi)$  is the Palm version of a stationary  $(Y, \eta)$ .*

An important ingredient in the proof of this theorem is the intrinsic characterization of Palm measures derived in [11].

## 5 The Poisson process and reversible shifts

We now turn to the other example mentioned in the introduction. This example concerns a stationary Poisson process  $\eta$  to which we add a point at the origin, thereby yielding the process  $\xi := \eta + \delta_0$ . In this setting, the new point is often referred to as a typical point of  $\xi$ .

For the Poisson process on the line ( $G = \mathbb{R}$ ), this is motivated by the fact that the intervals between the points of  $\xi$  have i.i.d. (exponential) lengths and thus if the origin is shifted to the  $n^{\text{th}}$  point on the right (or on the left) then the distribution of the process does not change:

$$\xi(T_n + \cdot) \stackrel{D}{=} \xi, \quad n \in \mathbb{Z}, \quad (5.1)$$

where  $T_0 := \pi_0(\xi) := 0$  and

$$T_n := \pi_n(\xi) := \begin{cases} n^{\text{th}} \text{ point on the right of the origin if } n > 0, \\ -n^{\text{th}} \text{ point on the left of the origin if } n < 0. \end{cases}$$

Since  $\xi$  looks distributionally the same from all its points, it is natural to say that the point at the origin is a typical point of  $\xi$ .

It is well known that on the line the typicality property (5.1) characterizes Palm versions  $\xi$  of stationary simple point processes  $\eta$  (but it is only in the Poisson case that the Palm version is of the form  $\eta + \delta_0$ ). Thus due to Theorem 4.2, (5.1) is equivalent to the origin being a typical location in the mass of  $\xi$  in the sense of Definition 4.1. Thus, – on the line, – calling the point at the origin a typical point is not only natural because of (5.1) but also consistent with Definition 4.1.

The property (5.1) is a more transparent definition of typicality than Definition 4.1, but it does not extend immediately beyond the line: if  $d > 1$  and we go out from the origin in any fixed direction then we will (a.s.) not hit a point of the Poisson process. One might conceive of mending this by ordering the points according to their distance from the origin, but this does not yield (5.1) as is clear from the following example.

**Example 5.1.** If  $\xi = \eta + \delta_0$  is the Palm version of a Poisson process  $\eta$  and we shift the origin to the point  $T$  that is closest to the origin, then the Poisson property is lost: the shifted process  $\xi(T + \cdot)$  is sure to have a point (the point at the old origin  $-T$ ) that is closer to the point at the origin than to any other point of  $\xi(T + \cdot)$ . This is not a property of  $\xi$  as the following argument shows.

The stationary Poisson process  $\eta$  need not have a point that is closer to the origin than to any other point of  $\eta$  since there is a positive probability that  $\eta$  has no point in the unit ball around the origin and that a bounded shell around that ball is covered by the balls of diameter  $\frac{1}{2}$  with centers at the points in the shell.

Thus for the Poisson process in the plane ( $G = \mathbb{R}^2$ ), – and in higher dimensions ( $G = \mathbb{R}^d$ ) and beyond, – there is no obvious motivation (save the analogy with the line) for calling the new point at the origin typical. However, adding that point to the stationary Poisson process yields its Palm version, and by Theorem 4.2 the origin is a typical location in the mass of the Palm version. Thus calling the point at the origin a typical point is again consistent with Definition 4.1.

Now although the property (5.1) does not extend immediately beyond the line, a generalization of (5.1) does. The key property of  $\pi_n$  defining  $T_n$  in (5.1) is that they are *reversible*: a measurable map  $\pi$  taking each  $\xi$  having a point at the origin to a point  $T = \pi(\xi)$  is reversible if it has a *reverse*  $\pi'$  such that

$$\pi'(\xi(T + \cdot)) = -T \quad \text{and} \quad \pi(\xi(T' + \cdot)) = -T' \quad \text{where} \quad T' = \pi'(\xi).$$

Above, the shift from the point at the origin to the  $n^{\text{th}}$  point on the right (or left) is reversed by shifting back to the  $n^{\text{th}}$  point on the left (or right). In Example 5.1 on the other hand, the shift to the closest point is not reversible because there can be more than one point having a particular point as their closest points. The following example of reversible  $\pi_n$  yielding a generalization of (5.1) is from [1].

**Example 5.2.** Let  $d = 2$  and consider  $\xi = \eta + \delta_0$  where  $\eta$  is a stationary Poisson process in  $\mathbb{R}^2$ . Link the points of  $\xi$  into a tree by defining the mother of each point as follows: place an interval of length one around the point parallel to the  $x$ -axis and send the interval off in the direction of the  $y$ -axis until it hits a point, let that point be the mother of the point we started from. Define the age-order of sisters by the order of their  $x$  coordinates. This procedure (see [1]) links the points into a one-ended tree such that each point has an ancestor with a younger sister.

Now put

$$\pi(\xi) = \begin{cases} \text{oldest daughter of 0, if 0 has a daughter,} \\ \text{oldest younger sister, if 0 has a younger sister but no daughter,} \\ \text{oldest younger sister of youngest ancestor who has a younger sister, else.} \end{cases}$$

This  $\pi$  is reversible with reverse  $\pi'$  defined by

$$\pi'(\xi) = \begin{cases} \text{mother of 0, if 0 has no older sister,} \\ \text{youngest older sister, if 0 has a daughterless youngest older sister,} \\ \text{last in youngest-daughter offspring-line of the youngest older sister, else.} \end{cases}$$

Put  $T_0 := \pi_0(\xi) := 0$  and recursively for  $n > 0$

$$\begin{aligned} T_n &:= \pi_n(\xi) := \pi(\xi(T_{n-1} + \cdot)) \\ T_{-n} &:= \pi_{-n}(\xi) := \pi'(\xi(T_{-(n-1)} + \cdot)). \end{aligned}$$

With this enumeration of the points of  $\xi$  the typicality property (5.1) holds, see [1].

For  $d > 2$  the same approach works to establish (5.1). In that case place a  $d - 1$  dimensional unit ball around each point and send the ball off in the  $d^{\text{th}}$  dimension until it hits a point. When  $d = 3$ , this again strings up all the points of  $\xi$  into the integer line. However when  $d > 3$ , this yields an infinite forest of trees, and the tree containing the point at the origin only strings up a subset of the points, see [1].

More sophisticated tree constructions can be found in [4] and [14]. In particular, the points can be linked into a single tree in all dimensions. And this is true not only for the Poisson process but for Palm versions of arbitrary stationary aperiodic simple point processes in  $\mathbb{R}^d$ .

## 6 Simple point processes and point-stationarity

The property (5.1) is a well known characterization of Palm versions  $\xi$  of stationary simple point processes on the line. When a random element  $X$  is involved and  $(X, \xi)$  is the Palm version of a stationary pair then the characterization reads as follows (recall that  $T_n^{-1} = -T_n$  is the group inverse of  $T_n$ ):

$$T_n^{-1}(X, \xi) \stackrel{D}{=} (X, \xi), \quad n \in \mathbb{Z}.$$

This is implied by the following property,

$$T^{-1}(X, \xi) \stackrel{D}{=} (X, \xi) \text{ for all } T = \pi(\xi) \text{ where } \pi \text{ is reversible,} \quad (6.1)$$

which is in turn implied by the following property,

$$T^{-1}(X, \xi) \stackrel{D}{=} (X, \xi) \text{ for all } T = \pi(X, \xi) \text{ where } \pi \text{ is reversible;} \quad (6.2)$$

here  $\pi$  *reversible* means that  $\pi$  has a *reverse*  $\pi'$  such that  $\pi'(T^{-1}(X, \xi)) = T^{-1}$  and  $\pi(T'^{-1}(X, \xi)) = T'^{-1}$  where  $T' = \pi'(X, \xi)$ .

The latter two properties are not restricted to the line, as we saw in Example 5.2. In [2] and [3] the property (6.1) is used to define *point-stationarity*, a precursor of mass-stationarity. There it is proved, for simple point processes on Abelian  $G$ , (i) that point-stationarity characterizes Palm versions of stationary pairs, (ii) that (6.1) can be replaced by (6.2), and (iii) that in (6.1) it suffices to consider  $\pi$  such that  $\pi' = \pi$  (such  $\pi$  are said to induce a *matching*).

Point-stationarity was introduced earlier in [12] (see also [13]) for simple point processes on  $G = \mathbb{R}^d$ , but the definition there was more cumbersome, involving *stationary independent backgrounds*: a random element  $Z$  (possibly defined on an extension of the underlying probability space) is a stationary independent background for  $(X, \xi)$  if

- (i)  $Z$  takes values in a measurable space on which  $G$  acts measurably, and
- (ii)  $Z$  is stationary and independent of  $(X, \xi)$ .

In [12]  $\xi$  is a simple point process on  $\mathbb{R}^d$  and the pair  $(X, \xi)$  is called point-stationary if for all stationary independent backgrounds  $Z$ ,

$$T^{-1}((Z, X), \xi) \stackrel{D}{=} ((Z, X), \xi) \text{ for all } T = \pi((Z, X), \xi) \text{ where } \pi \text{ is reversible.} \quad (6.3)$$

This property was proved to characterize Palm versions  $(X, \xi)$  of stationary pairs and to be equivalent to what later became the definition of mass-stationarity. The proof of the fact that (6.3) implies (3.1) with  $C = [0, 1]^d$  is sketched in the following example. The result for  $C = [0, h]^d$  is obtained in the same way, and the result for relatively compact  $C$  then follows by a simple conditioning argument.

**Example 6.1.** Consider  $G = \mathbb{R}^d$ . Let  $U_C$  be uniform on  $C = [0, 1]^d$  and  $U$  be uniform on  $[0, 1]$ . Let  $U_C$  and  $U$  be independent and independent of  $(X, \xi)$ . Put  $Z = (U_C^{-1}\mathbb{Z}, U)$  and let shifts leave  $U$  intact. Let  $\pi_n(Z, \xi)$  be the  $n^{\text{th}}$  point of  $\xi$  after the point at the origin in the circular lexicographic ordering of the points in the set  $U_C^{-1}C$ . These  $\pi_n$  are reversible (with  $\pi'_n$  obtained from the reversal of the lexicographic ordering), and so is the mapping  $\pi$  defined by

$$\pi((Z, X), \xi) := \pi(Z, \xi) := \pi_{[U\xi(U_C^{-1}C)]}(Z, \xi).$$

Now  $V_C := \pi(Z, \xi)$  has the conditional distribution  $\xi(\cdot \mid U_C^{-1}C)$  given  $((Z, X), \xi)$ , and thus also given  $(X, \xi, U_C)$  since  $U_C$  and  $Z$  are measurable functions of each other. Thus (6.3) implies (3.1) for this particular set  $C$ .

The results mentioned above together with Theorem 4.1 yield the following theorem.

**Theorem 6.2.** *Let  $\xi$  be a simple point process on a locally compact Abelian  $G$  having a point at the origin. Allow the distributions of  $(X, \xi)$  and  $(Y, \eta)$  to be only  $\sigma$ -finite. Then the following claims are equivalent:*

- (a) *the pair  $(X, \xi)$  is mass-stationary,*



- (b) the pair  $(X, \xi)$  is the Palm version of a stationary  $(Y, \eta)$ ,
- (c) the pair  $(X, \xi)$  is point-stationary,
- (d) the property (6.1) holds with  $\pi$  restricted to be its own reverse (matching),
- (e) the property (6.2) holds,
- (f) the property (6.3) holds for all stationary independent backgrounds  $Z$ .

*Proof.* The only claim that has not been proved is that (f) can be added to the equivalences (a) through (e) in the general Abelian case. For that purpose assume that (b) holds and let  $Z$  be stationary and independent of  $(X, \xi)$  and  $(Y, \eta)$ . Then  $((Z, X), \xi)$  is the Palm version of  $((Z, Y), \eta)$  and the equivalence of (b) and (e) yields (f). Conversely, (e) follows from (f).  $\square$

## 7 Measure preserving allocations

For a measurable map  $\pi$  taking a random measure  $\xi$  to a location  $\pi(\xi)$  in  $G$ , define the associated  $\xi$ -allocation  $\tau$  by

$$\tau(t) = \tau_\xi(t) = t\pi(t^{-1}\xi), \quad t \in G.$$

Similarly, for a measurable map  $\pi$  taking  $(X, \xi)$  to a location  $\pi(X, \xi)$  in  $G$ , define the associated  $(X, \xi)$ -allocation  $\tau$  by

$$\tau(t) = \tau_{(X, \xi)}(t) = t\pi(t^{-1}(X, \xi)), \quad t \in G.$$

The  $\pi$  in the definition of reversibility above is defined for simple point processes  $\xi$  having a point at the origin. If we define  $\pi$  for simple point processes  $\xi$  *not* having a point at the origin by  $\pi(\xi) = 0$  and  $\pi(X, \xi) = 0$ , respectively, then  $\pi$  is reversible if and only if the associated  $\tau$  is a bijection. The bijectivity of  $\tau$  is further equivalent to  $\tau$  *preserving* the measure  $\xi$ , that is, for each fixed value of  $\xi$  the image measure of  $\xi$  under  $\tau$  is  $\xi$  itself:

$$\xi(\{s \in G : \tau(s) \in A\}) = \xi(A), \quad A \in \mathcal{G},$$

or in probabilistic notation,

$$\xi(\tau \in \cdot) = \xi.$$

Preservation and bijectivity are, however, only equivalent if we restrict to the simple point process case. Preservation (rather than reversibility/bijectivity) turns out to be the property that is essential for going beyond simple point processes.

Say that  $\pi$  is *preserving* if the associated  $\tau$  preserves  $\xi$ . In [8] it is shown that the following analogue of (6.1),

$$T^{-1}(X, \xi) \stackrel{D}{=} (X, \xi) \text{ for all } T = \pi(\xi) \text{ where } \pi \text{ is preserving,} \quad (7.1)$$

does *not* suffice to characterize the Palm versions of a stationary random measures with point masses of different positive sizes since an allocation cannot split a positive point mass. Neither does (7.1) with  $T = \pi(X, \xi)$  for the same reason. One might therefore want to restrict attention to *diffuse* random measures, that is, random measure with no positive point masses. It is not known yet whether (7.1) does suffice to characterize Palm versions in the diffuse case. However, this is true when  $G = \mathbb{R}^d$  if stationary independent backgrounds are allowed. The following result is from the forthcoming paper [10].

**Theorem 7.1.** *Let  $\xi$  be a diffuse random measure on  $\mathbb{R}^d$  having the origin in its support. Then the following claims are equivalent:*

- (a) *the pair  $(X, \xi)$  is mass-stationary,*
- (b) *for all stationary independent backgrounds  $Z$ ,*

$$T^{-1}((Z, X), \xi) \stackrel{D}{=} ((Z, X), \xi) \text{ for all } T = \pi(Z, \xi) \text{ where } \pi \text{ is preserving,}$$

- (c) *for all stationary independent backgrounds  $Z$ ,*

$$T^{-1}((Z, X), \xi) \stackrel{D}{=} ((Z, X), \xi) \text{ for all } T = \pi((Z, X), \xi) \text{ where } \pi \text{ is preserving.}$$

## 8 Cox and Bernoulli randomizations

Stationary independent backgrounds constitute a certain kind of randomization. Another kind of randomization, a Cox randomization, yields a full characterization of mass-stationarity in the Abelian case as we now explain.

Consider a *Cox process* driven by  $(X, \xi)$ , that is, an integer-valued point process which conditionally on  $(X, \xi)$  is a Poisson process with intensity measure  $\xi$ . Intuitively, the Cox process can be thought of as representing the mass of  $\xi$  through a collection of points placed independently at typical locations in the mass of  $\xi$ . Thus if  $(X, \xi)$  is mass-stationary (if the origin is a typical location for  $X$  in the mass of  $\xi$ ) and we add an extra point at the origin to the Cox process, then the points of that *modified* Cox process  $N$  are *all* at typical locations in the mass of  $\xi$ .

It turns out that mass-stationarity reduces to mass-stationarity with respect to this modified Cox process; for proof see [9].

**Theorem 8.1.** *Let  $\xi$  be a random measure on an Abelian  $G$ . Then the following claims are equivalent:*

- (a) *the pair  $(X, \xi)$  is mass-stationary,*
- (b) *the pair  $(X, N)$  is mass-stationary,*
- (c) *the pair  $((X, \xi), N)$  is mass-stationary.*

In the diffuse case, the modified Cox process  $N$  is a simple point process and mass-stationarity reduces to point-stationarity by Theorem 6.2:

**Corollary 8.2.** *Let  $\xi$  be a diffuse random measure on an Abelian  $G$ . Then the following claims are equivalent:*

- (a) *the pair  $(X, \xi)$  is mass-stationary,*
- (b) *the pair  $(X, N)$  is point-stationary,*
- (c) *the pair  $((X, \xi), N)$  is point-stationary.*

Due to this result the various reversible shifts that are known for simple point processes can now be applied to diffuse random measures through the modified Cox process  $N$ .

Yet another kind of randomization, a Bernoulli randomization, works in the discrete case. A *Bernoulli transport* refers to a randomized allocation rule  $\tau$  that allows staying at a location  $s$  with a probability  $p(s)$  depending on  $s^{-1}(X, \xi)$  and otherwise chooses another location according to a (non-randomized) allocation rule. Call the associated  $\pi$  *Bernoulli*. This makes it possible to split discrete point-masses. The following result is from [9].

**Theorem 8.3.** *Let  $\xi$  be a discrete random measure on an Abelian  $G$ . Then  $(X, \xi)$  is mass-stationary if and only if*

$$T^{-1}(X, \xi) \stackrel{D}{=} (X, \xi)$$

for all  $T = \pi(\xi)$  where  $\pi$  is preserving and Bernoulli.

## 9 Mass-stationarity through bounded invariant kernels

We conclude with a more analytical characterization of mass-stationarity. A kernel  $K_{(X, \xi)}$  from  $G$  to  $G$  is *preserving* if

$$\int K_{(X, \xi)}(s, A) \xi(ds) = \xi(A), \quad A \in \mathcal{G},$$

and *invariant* if

$$K_{(X, \xi)}(t, A) = K_{t^{-1}(X, \xi)}(0, t^{-1}A), \quad t \in G, A \in \mathcal{G}.$$

Note that if  $\tau$  is a preserving allocation then the kernel defined by

$$K_{(X, \xi)}(t, A) = 1_A(\tau(t))$$

is preserving and invariant. It is also Markovian and therefore bounded.

In the Abelian case the following result is from [8]. For the general case, which can be handled as in Section 3.8 of [7], we need the modular function  $\Delta : G \rightarrow (0, \infty)$  of  $G$  ( $\Delta \equiv 1$  in the Abelian case).

**Theorem 9.1.** *The pair  $(X, \xi)$  is mass-stationary if and only if for all preserving invariant bounded kernels  $K$  and all nonnegative measurable functions  $f$ ,*

$$\mathbb{E} \left[ \int f(s^{-1}(X, \xi)) \Delta(s^{-1}) K_{(X, \xi)}(0, ds) \right] = \mathbb{E}[f(X, \xi)]. \quad (9.1)$$

If  $G$  is Abelian and  $K_{(X, \xi)}$  is Markovian then (9.1) means that

$$T^{-1}(X, \xi) \stackrel{D}{=} (X, \xi)$$

where  $T$  has conditional distribution  $K_{(X, \xi)}(0, \cdot)$  given  $(X, \xi)$ . It is not known yet whether ‘bounded’ in the theorem can be replaced by ‘Markovian’.

Theorem 9.1 and Theorem 4.2 yield the following extension of Theorem 3.2(b) to the locally compact case.

**Theorem 9.2.** *The pair  $(X, \lambda)$  is mass-stationary (that is, the origin is a typical location for  $X$ ) if and only if  $X$  is stationary.*

*Proof.* Suppose  $X$  is stationary. Then so is  $(X, \lambda)$ . A stationary  $(X, \lambda)$  is the Palm version of itself. Thus Theorem 4.2 yields the fact that  $(X, \lambda)$  is mass-stationary. Conversely, assume that  $(X, \lambda)$  is mass-stationary. Fix an arbitrary  $t \in G$  and let  $K_{(X, \lambda)}$  be the invariant kernel with  $K_{(X, \lambda)}(0, A) = \Delta(t)1_A(t)$ . This kernel is preserving and from (9.1) we obtain that  $\mathbb{E}[f(t^{-1}(X, \lambda))] = \mathbb{E}[f(X, \lambda)]$ . Since this holds for all nonnegative measurable  $f$  it holds in particular for  $f$  that are constant in the second argument and thus  $X \stackrel{D}{=} Y$ . Hence  $X$  is stationary.  $\square$

Theorem 9.2 shows that mass-stationarity is a generalization of the concept of stationarity.

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